

Propagator with Positive Cosmological Constant in the 3D Euclidian Quantum Gravity Toy Model

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Abstract

We study the propagator on a single tetrahedron in a three dimensional toy model of quantum gravity with positive cosmological constant. The cosmological constant is included in the model via q-deformation of the spatial symmetry algebra, that is, we use the Turaev-Viro amplitude. The expected repulsive effect of dark energy is recovered in numerical and analytic calculations of the propagator at large scales comparable to the infrared cutoff. However, due to the simplicity of the model we do not obtain the exact Newton limit of the propagator. This is a first step toward the similar calculation in the full 3+1 dimensional theory with larger numbers of simplicies.

I. INTRODUCTION

This paper extends work [1–3] evaluating quantum gravity two-point functions on a single tetrahedron in three euclidean dimensions to the case with positive cosmological constant. At long distance scales the two point function for a quantum field theory gives the Newton force associated to that theory’s gauge boson. In our toy model, since we only have one tetrahedron and we peak our state around an equilateral configuration of that tetrahedron we do not expect to reproduce the exact Newton limit of 3D gravity with a cosmological constant. However, we still do find an asymptotically repulsive force associated to the dark energy.

We take the inclusion of the cosmological constant to correspond to a deformation of the $SU(2)$ spatial rotation symmetry. This quantum gravity toy model is known as the q-deformed Ponzano-Regge model or the Turaev-Viro model. We expect two key differences in the propagator from the Ponzano-Regge model. First, in the case where the tetrahedra is large compared with the infrared cutoff imposed by the cosmological constant, the sums will be cut off by the deformation and not the triangular inequalities. This feature of the quantum algebra is essential in four dimensions where we cannot escape so-called “bubble” configurations of 2-complexes, without an infrared cutoff these transition amplitudes would diverge. Second, in the case where the size of the tetrahedron is smaller than the cutoff, the modification will simply affect the asymptotics of the two point function via the addition of a volume term to the Regge action.

In this paper we first compute the Turaev-Viro propagator numerically in Section II. Then we compare the numerical computations with an analytic calculation of the propagator asymptotics in Section III. These two calculations are found to match strongly in an easily computable regime, that is for an unrealistically large cosmological constant. However, the agreement is expected to persist as the cosmological constant is taken smaller.

II. METHOD AND RESULTS

This paper studies the correlator between two edges on a single tetrahedron. We fix four edges around the tetrahedron to j_0 which can be thought of as a time, thus j_1 and j_2 are lengths on two different time slices of the spacetime, see Figure 1. We will study the modulus of the propagator $|\mathcal{P}|$ as a function of the distance j_0 and the infrared cutoff j_{\max} .

Because in the 3D case all of the calculations can be done in a gauge where they all give zero, we clearly have an issue of gauge choice. We follow Speziale in by picking a Coulomb-like gauge where

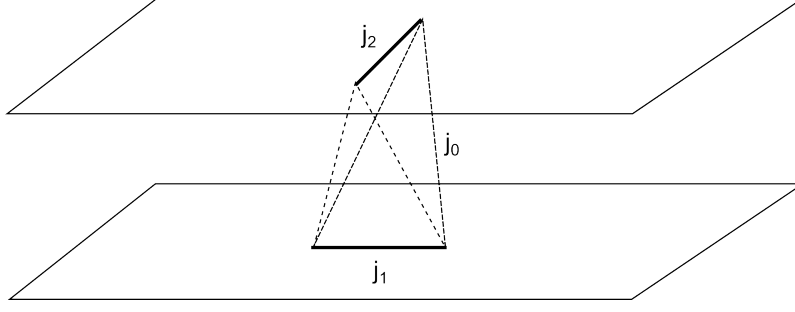


Figure 1. Depiction of the tetrahedron state used in this paper. The the four edges of the tetrahedron that go between the boundary slices are fixed to a value of j_0 and can be thought to correspond to the time. We then study the two point function between the length operators acting on each of the two edges on the slices.

the field operators have non-trivial projections along a bone (edge of a triangle in the triangulation) [1]. This choice will produce a calculation similar to the one that can be done in the 4D spinfoam model [4]. We choose our operator notations different than Speziale, but in line with more recent work in quantum gravity correlation functions [5, 6]. The two point function is:

$$\mathcal{P}_{nm}^{abcd} := \frac{\langle W | \mathcal{P}_n^{ab} \mathcal{P}_m^{cd} | \psi_\Sigma \rangle_q}{\langle W | \psi_\Sigma \rangle_q}, \quad (1)$$

where instead of the usual metric field insertions we have the operators perturbed around flat space.

$$\begin{aligned} \mathcal{P}_n^{ab} |s\rangle &:= l_{a,n}^\mu l_{b,n}^\nu h_{\mu\nu}(x) |s\rangle \\ &= l_{a,n}^\mu l_{b,n}^\nu (g_{\mu\nu}(x) - \delta_{\mu\nu}) |s\rangle \\ &= (l_p^2 G_n^{ab} - l_p^2 C_q^2(j_0)) |s\rangle \end{aligned}$$

Above $G_n^{ab} := \vec{L}_n^a \cdot \vec{L}_n^b$ are the Penrose operators acting on links that go between nodes a and n , and b and n in the spin network state $|s\rangle$, C_q is the $SU_q(2)$ casimir defined as $C_q(j) := \sqrt{[j][j+1] + 1/4}$, W is the Turaev-Viro transition amplitude and ψ_Σ is a boundary state peaked on an equilateral tetrahedron. For a tetrahedron the spin network state is also a tetrahedron dual to the original one where there are nodes on all the faces. We would like to choose the node labels such that we are calculating the correlator between the two edges labeled by j_1 and j_2 . For instance, we could label the four nodes such that face 1 opposes face 3 and face 2 opposes face 4. Then we would be interested in \mathcal{P}_{34}^{1122} . We calculate this by evaluating equation 1 with the q-deformed state and

transition amplitude, ie.:

$$\mathcal{P}_{34}^{1122} = \frac{1}{j_0^4 l_p^4 \mathcal{N}} \sum_{j_1, j_2}^{2j_0} W_q(j_1, j_2; j_0) \psi_\Sigma^q(j_1, j_2; j_{\max}) l_p^4 (C_q^2(j_1) - C_q^2(j_0)) (C_q^2(j_2) - C_q^2(j_0)), \quad (2)$$

where we have from the Tuarev-Viro model that the transition amplitude for a single tetrahedron is just a quantum 6j-symbol:

$$W_q(j_1, j_2; j_0) = \left\{ \begin{array}{ccc} j_1 & j_0 & j_0 \\ j_2 & j_0 & j_0 \end{array} \right\}_q,$$

we take the deformation parameter to be a root of unity $q = \exp(i\sqrt{\Lambda}\hbar G)$ where Λ is the cosmological constant that fixes the infrared cutoff j_{\max} as $\sqrt{\Lambda} = \pi/(2j_{\max} + 1)$ [7]. The boundary state is taken to be:

$$\psi_\Sigma^q = \frac{1}{N} \exp \left(-\frac{\alpha}{2} \sum_i (j_i - j_0)^2 + i\theta \sum_i \left(j_i + \frac{1}{2} \right) - i \frac{\sqrt{\Lambda} j_0^2}{12\sqrt{2}} \sum_i (j_i - j_0) \right).$$

The factor with Λ in this state is included so that the asymptotics do not have oscillatory terms in the limit as $j_0 \rightarrow \infty$ that come from the addition of the cosmological constant term to the Regge action. Numerical evaluation of equation 2 gives the main result of the paper, see figure 2. Here we can see that at small scales $j_0 \approx 1$ we have the same quantum deviations as Speziale from the Newtonian limit. In the range where we are not too close to the cutoff and not too small lengths ie. $1 \ll j_0 \ll j_{\max}/2$ we have behavior similar to the Newtonian limit without cosmological constant $|\mathcal{P}| \approx 3/2j_0$. Lastly, when the spins get close to the infrared cutoff we have a repulsion representative of the repulsive force of dark energy.

III. DISCUSSION AND ANALYSIS

We would like to analyze analytically the asymptotics of this toy model to see if we can match the numerical result. The asymptotics of $\{6j\}_q$ symbols are given by a cosine of the Regge action with a volume term [8, 9].

$$\{6j\}_q \sim \frac{2\pi \cos \left(S_R[j_e] - \sqrt{\Lambda} \text{Vol}(\tau_S) + \frac{\pi}{4} \right)}{r^{3/2} G^{1/4}}$$

The volume term is the volume of a spherical tetrahedron. We will approximate this as a flat tetrahedron and expand the volume in δj_1 and δj_2 as:

$$\text{Vol}(\tau_S) \approx \text{Vol}(\tau_E) \approx \frac{(1+2j_0)^3}{48\sqrt{2}} + \frac{(1+2j_0)^2}{48\sqrt{2}} (\delta j_1 + \delta j_2) - \frac{7(1+2j_0)}{96\sqrt{2}} (\delta j_1^2 + \delta j_2^2) - \frac{(1+2j_0)\delta j_1 \delta j_2}{48\sqrt{2}}.$$

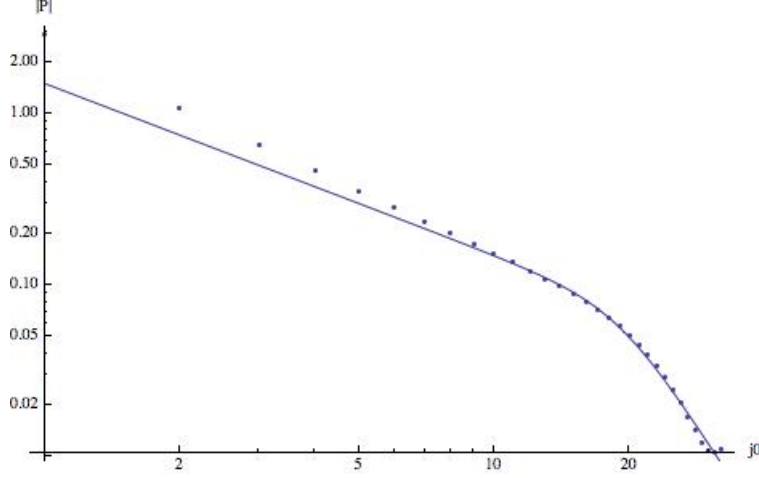


Figure 2. Plot of the q-deformed propagator calculated for $j_{\max} = 65$ and various values of j_0 . The asymptotic behavior is clearly modified by the presence of the cosmological constant via a repulsion at large distances. This repulsion is well captured in the analytic calculation of the curve in the figure. Additionally, in the intermediate regime where $1 \ll j_0 \ll j_{\max}/2$ we have agreement with the Speziale asymptotics ie. $|\mathcal{P}| \sim 3/2j_0$.

The factor of $2\pi/r^{3/2}G^{1/4}$ is approximately independent of j_1 and j_2 therefore it will cancel with the same factor in the normalization. Also in the large spin limit the difference of the Casimir operators will go like $C^2(j_1) - C^2(j_0) \sim 2j_0\delta j_1$. We then arrive at an asymptotic formula for the propagator:

$$|\mathcal{P}| \sim \frac{4}{j_0^2 \mathcal{N}} \sum_{j_1, j_2}^{2j_0} \delta j_1 \delta j_2 \exp \left[-iS_R[j_e] + \sqrt{\Lambda} i \text{Vol}(\tau_E) - i\frac{\pi}{4} - \frac{\alpha}{2} \sum_i \delta j_i^2 + i\theta \sum_i \left(j_i + \frac{1}{2} \right) - \frac{i\sqrt{\Lambda} j_0^2}{12\sqrt{2}} \sum_i \delta j_i \right],$$

where we have omitted the other exponential term that is rapidly oscillating in j_1 and j_2 . Expanding action and the volume terms around the equilateral configuration, and canceling factors independent of j_1 and j_2 with the same ones in the normalization we obtain:

$$|\mathcal{P}| \sim \frac{4}{j_0^2 \mathcal{N}} \sum_{j_1, j_2}^{2j_0} \delta j_1 \delta j_2 \exp \left[-\frac{i\sqrt{\Lambda}}{2} \left(\frac{7j_0}{24\sqrt{2}} (\delta j_1^2 + \delta j_2^2) + \frac{j_0 \delta j_1 \delta j_2}{12\sqrt{2}} \right) - \frac{i}{2} \sum_{i,k} G_{ik} \delta j_i \delta j_k - \frac{\alpha}{2} \sum_{i=1}^2 \delta j_i^2 \right],$$

where $G_{ik} = \partial\theta_i/\partial j_k|_{j_e=j_0}$ is the matrix of derivatives of the dihedral angles with respect to the edge lengths evaluated for an equilateral tetrahedron. Passing to continuous variables $z = \delta j_1$ and $dz = dj_1$ we obtain the gaussian integral:

$$\mathcal{P} \sim \frac{4}{j_0^2 \mathcal{N}} \int d^2 z \, z_1 z_2 \exp \left[-\frac{1}{2} z_i A_{ik} z_k \right] = \frac{4}{j_0^2} (A^{-1})_{12},$$

where A is the matrix of coefficients of $\delta j_i \delta j_k$:

$$A_{ik} = \frac{4}{3j_0} \begin{pmatrix} 1 + i\frac{\sqrt{2}}{4} & i\frac{3\sqrt{2}}{4} \\ i\frac{3\sqrt{2}}{4} & 1 + i\frac{\sqrt{2}}{4} \end{pmatrix} + \frac{i\sqrt{\Lambda}(j_0 + 1/2)}{24\sqrt{2}} \begin{pmatrix} 7 & 1 \\ 1 & 7 \end{pmatrix},$$

where we have set $\alpha = 4/3j_0$ which is compatible with the requirement that the state be peaked on the intrinsic and extrinsic geometry [1]. From this, we have the full expression for the asymptotics of the propagator modulus:

$$|\mathcal{P}| \sim \frac{4\sqrt{6}(96 + J_\Lambda)^2}{j_0} \frac{1}{\sqrt{2^{17} \cdot 3 + J_\Lambda(2^{13} + J_\Lambda(128 + J_\Lambda(32 + 3J_\Lambda)))}}$$

where $J_\Lambda := j_0(1 + 2j_0)\sqrt{\Lambda}$. The plot of this analytic expression versus the numerical calculation is shown in Figure 2 and the agreement is seen to be very good. If we expand this expression about $\Lambda = 0$ it becomes more clear that a repulsion is the dominating correction to the Speziale result.

$$|\mathcal{P}| \sim \frac{3}{2j_0} - \frac{1}{1536}(j_0 + 1/2)^2 j_0 \Lambda + O(\Lambda^{3/2})$$

We obtain an expression with a term proportional to the cosmological constant times the volume of the tetrahedron. While this differs from what we might expect from the Newton law, we should remember that this is a simplified model where there is only one simplex that spans the cosmological distance, and it is peaked on an equilateral configuration. Further investigations could attempt a similar calculation where there was a small length scale introduced to make an elongated tetrahedron instead of an equilateral one, which would better capture the true propagator physics. Despite these caveats we still observe the characteristic repulsion of a positive cosmological constant.

IV. SUMMARY AND CONCLUSIONS

We have shown that the inclusion of a cosmological constant in a 3D euclidean toy model of quantum gravity on a single tetrahedron reproduces the expected qualitative behavior near the infrared cutoff. Namely, we expect to have an additional repulsive force on large distance scales. The effect is reproduced in both numerical calculations and an analytic evaluation of the propagator asymptotics. This work also strengthens the argument for the interpretation of the cosmological constant as a deformation parameter in the theory. However, as mentioned in the introduction, we do not produce the exact form of the Newton propagator, which is thought to be due to the simplicity of the model.

This result can be taken as a first step toward performing a similar calculation in 3+1 dimensions. Given that there are already asymptotic calculations of the propagator [4] and the $SU(2)$ symmetry is still the relevant one this may not be too difficult for one 4-simplex. Another direction this work could be extended is to multiple tetrahedra. The numerical computations become significantly more difficult as the number of simplicies increase, however one could probably check the analytic results to see if the repulsion persists.

V. ACKNOWLEDGEMENTS

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